Math in Industry
Siemens Health Care
Slugging Along Tube
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• Abouali, Mohammad (San Diego State University)
• Breward, Chris (Oxford University)
• Chan, Ian (University of British Columbia)
• Ellis, Andrew (Oxford University)
• Fehribach, Joseph D. (Worcester Polytechnic Institute)
• Gratton, Michael (Northwestern University)
• Matusik, Katarzyna (University of California, San Diego)
OutLine:

- Problem Definition
- Incompressible Flow
- Compressible Flow
  - Dry Chamber:
    - One Slug
    - Two Slugs
  - Wet Chamber:
    - Bubble Formation and One Slug
- Recommendations
Problem Definition:

- System Description:
  - A liquid slug of length $l$ is driven into a cylindrical reaction chamber by a flux of gas, which then vents to atmosphere

- Inquiries:
  - What is the pumping protocol for the delivery of a slug down a prescribed distance $x$ into the reaction chamber?
  - What is the effect of having more than one slug in the tube at one time?
  - What is the cause of foam formation within the reaction chamber, and how may this be avoided?
Part I: Incompressible Flow
Schematic View (One Slug):

\[ L_s \]

\[ A_s \]

\[ V_0 = 2500\mu l \]

\[ P \]

\[ \ell \]

\[ A_t \]

\[ x \]

\[ A_s = 3.14 \times 10^{-6} m^2; \quad \ell = 0.5 \sim 5 cm; \quad A_t = 1.96 \times 10^{-7} m^2 \]

\[ P_0 = P_{atm} \]

Vent
In incompressible case, everything is rather easy.

The relation between the plunger and the slug is:

\[ x = \frac{A_s}{A_t} y \]

So, if we assume \( A_s = A_t \), we need a 6 m long syringe to push the slug 6 m down the tube.
Part II: Compressible Flow
Dry Chamber - One Slug
Schematic View (One Slug):

\[ V = A_t x + A_s (L_s - y) \]

- \( L_s \)
- \( A_s \)
- \( V_0 = 2500 \mu l \)
- \( P \)
- \( \ell \)
- \( A_t \)
- \( P_0 = P_{atm} \)

- \( A_s = 3.14 \times 10^{-6} \ m^2 \)
- \( \ell = 0.5 \sim 5 \ cm \)
- \( A_t = 1.96 \times 10^{-7} \ m^2 \)
Model for single slug:

- We have:
  \[(P - P_0)A_t = m\ddot{x} + c\dot{x}\]

- Ideal Gas Law at constant temperature (Boyle’s Law):
  \[PV = P_0V_0\]
  \[V_0 = 2500\mu l\]

- Remember:
  \[V = A_tx + A_s(L_s - y)\]

- Finally:
  \[P_0A_t\left(\frac{V_0}{A_tx + A_s(L_s - y)} - 1\right) = \rho_wA_t\ell\ddot{x} + \frac{8\pi\mu\ell\dot{x}}{m}\]
Finding ‘c’:

- We have:
  \[ \Delta P = P_b - P_c - P_0 + P_c \]
  \[ \ell_p = \ell - 2r_w \]

- From Pipe Flow:
  \[ \bar{u} = \frac{\Delta P}{8\mu\ell_p} r_w^2 = \dot{x} \]

- Using force balance:
  \[ c \dot{x} = A_t (P_b - P_0) = A_t \Delta P \]
  \[ c \frac{\Delta P}{8\mu\ell_p} r_w^2 = \Delta P \pi r_w^2 \]
  \[ c = 8\pi\mu\ell_p \]
Non-Dimensional Form:

- Length Scales:
  \[ x \Rightarrow L_t x \]
  \[ y \Rightarrow \frac{A_t L_t}{A_s} y \]

- The non-dimensional form would be:
  \[
  \left( \frac{1}{1 + \frac{\delta(x - y)}{1 + \delta(x - y)}} - 1 \right) = \alpha \ddot{x} + \beta \dot{x}
  \]

- Where:
  \[
  \alpha = \frac{\rho_w \ell L_t}{P_0 T^2}; \quad \beta = \frac{8\pi \mu \ell L_t}{A_t P_0 T}; \quad \delta = \frac{A_t L_t}{A_s L_s}
  \]
Results:
Dry Chamber - Two Slugs
Schematic View of Two Slug:

\[ A_s = 3.14 \times 10^{-6} \text{ m}^2; \quad \ell = 0.5 \sim 5 \text{cm}; \quad A_t = 1.96 \times 10^{-7} \text{ m}^2 \]
Model for Two Slugs:

- We have:
  \[
  \begin{align*}
  A_t(P_1 - P_2) &= m_1 \ddot{x} + c_1 \dot{x} \\
  A_t(P_2 - P_0) &= m_2 \ddot{x} + c_2 \dot{x}
  \end{align*}
  \]

- Choosing the Scale:
  \[
  x \Rightarrow L_t x; \quad y \Rightarrow \frac{A_t L_t}{A_s} y
  \]

- We have:
  \[
  \begin{align*}
  \left[ \frac{1}{\delta_1(x_1 - y) + 1} - \frac{\delta_3}{(x_2 - x_1) - \delta_2} \right]
  &= \alpha_1 \ddot{x}_1 + \beta_1 \dot{x}_1 \\
  \left[ \frac{\delta_3}{(x_2 - x_1) - \delta_2} - 1 \right]
  &= \alpha_2 \ddot{x}_2 + \beta_2 \dot{x}_2
  \end{align*}
  \]

- Where:
  \[
  \begin{align*}
  \alpha_i &= \frac{\rho_w \ell_i L_t}{P_0 T^2} ; \quad \beta_i = \frac{8 \pi \mu \ell_i L_t}{A_t P_0 T} ; \quad \delta_1 = \frac{A_t L_t}{A_s L_s} ; \quad \delta_2 = \frac{\ell_1}{L_t} ; \quad \delta_3 = \frac{d_1}{L_t}
  \end{align*}
  \]
Case I: (Sub 1→Red; Sub 2→Blue)

- We have: $l_1 > l_2$
Case II: (Sub 1 $\rightarrow$ Red; Sub 2 $\rightarrow$ Blue)

- We have: $l_1 = l_2$
Case III: (Sub 1→Red; Sub 2→Blue)

- We have: \( l_1 < l_2 \)
The Gap:

Changes in the distance between the two slugs are:
Wet Chamber: Bubble Formation & One Slug Model
Schematic View of Wet Chamber:

\[ A_s = 3.14 \times 10^{-6} \text{ m}^2; \quad \ell = 0.5 \sim 5 \text{ cm}; \quad A_t = 1.96 \times 10^{-7} \text{ m}^2 \]
Bubble Formation

- This is the modified equation for one Slug with a wet chamber.

\[
\left( \frac{P_b}{P_0} - \frac{P_f}{P_0} \right) = \alpha \ddot{x} + \beta \dot{x}
\]

\[
\frac{P_f}{P_0} = \min\left( \frac{V_f^0}{A_p(L_p - (x_n + l_n))}, \frac{P^*}{P_0} \right)
\]
Results:

Back Pressure

Pressure vs. Time graph

$P_c$
Results: (cont.)
Control Problem:

To have an idea of how to control the rate of bubbling, we need to consider:

- Bubbling due to the air in front of the slug,
- Bubbling due to the air pressure built up in the back.

\[ \dot{x} = \frac{(P_b - P_*) A_t}{c} \]

\[ P_0 = P_{\text{atm}} \]

\[ P_f = P_* \]
Back Pressure Problem:

\[
\frac{A_p L_p}{A_s L_s} < Y < \frac{A_p L_p}{A_s L_s} + L_p \left(1 - \frac{P_0}{P_*}\right)
\]
Part IV: Recommendations
Recommendations:

- Install air permeable membrane near reaction chamber to allow air to escape (but not liquid).
- Don’t push too hard
- Scheduling of the arrival time of the reagents.
Thank You