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Optimal Wear for a Laying Pipe

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Abstract

The optimal design of the Morgan Construction Companies' laying pipe is investigated. This crucial component of their rolling mill equipment suffers from rapid, uneven wear. During the 2003 Mathematical Problems in Industry Workshop, a team of ten investigators from five universities and one national laboratory developed a pipe design approach intended to minimize and evenly distribute wear.

1 Introduction

The Morgan Construction Company of Worcester Massachusetts designs and manufactures rolling mill equipment for the steel industry. One popular type of rolling mill starts with a large billet of hot steel and produces thin steel rod. During the rolling process, the initial billet, with cross sectional area on order of 180mm^2, is squeezed down to rod with cross sectional area on order of a few square millimeters. Since mass and momentum are conserved the initial slowly travelling billet will be travelling rapidly upon conversion to thin rod. In fact, at the last stage of the rolling process the rod attains speeds of up to 120m/s.[1] This rapidly travelling wire must be brought to

[1] This is about 268 miles per hour, roughly the speed at which Bob Hart drives.

rest in a controlled fashion for cooling and handling. This is the function of the so-called "Laying Head." The Morgan laying head is shown in Figure 1.

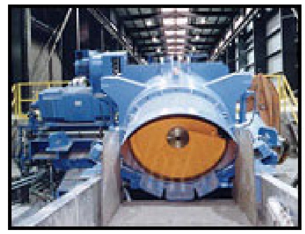

Figure 1: The Morgan Construction laying head. The photo on the left shows the laying head in action with hot coils of steel exiting the machine. The photo on the right shows a close up view of the laying head. Photos courtesy of Morgan Construction.

We note that the straight rod which enters the machine at 120m/s exits as stationary coiled rod and is conveyed to a packaging machine on a conveyor belt moving at walking speeds. In order to accomplish the transformation from rapidly moving straight rod to stationary coiled rod, the steel is shot through a curved pipe. An idealized pipe design is shown in Figure 2. The steel rod enters the straight portion of the pipe at the right side of Figure 2. The pipe is attached to the interior of a large cone and rotates rapidly about the axis defined by this initial straight portion of the pipe. Through collision with the pipe walls, the straight line motion of the wire is redirected to a direction perpendicular to this initial motion. Additionally, the rotation of the pipe induces angular acceleration in the wire. At the far end of the pipe, the wire is stationary, and has attained a circular shape, the plane of the circle being perpendicular to the original wire direction.

Collision of the wire with the pipe walls causes rapid uneven wear of the pipe. After only a few days of operation, this wear may lead to the wire breaking through the pipe. This is called a "cobble." Naturally, it is desirable to avoid this situation. Consequently, the pipe is routinely replaced, resulting in costly mill shutdowns. If the pipe lifetime could be extended by only a few days, tremendous savings would be realized. On the other hand, solutions to the wear problem which involving adding components to the laying head are not cost effective. A balanced solution is to optimize the pipe geometry to minimize and evenly distribute pipe wear.

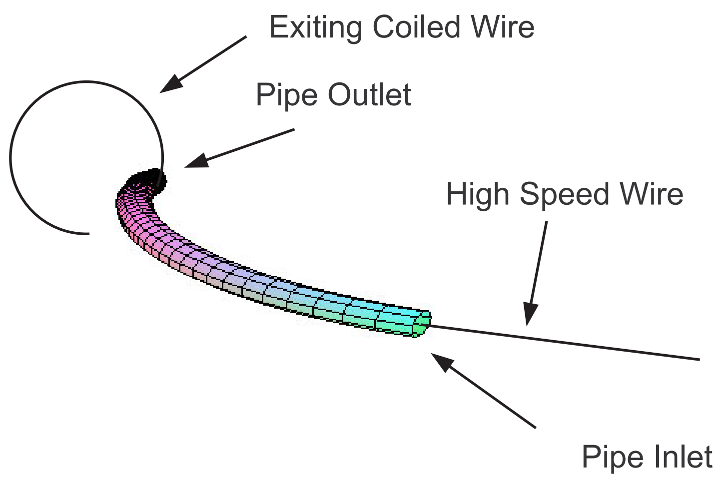

Figure 2: An idealized pipe design. Note that the pipe is contained within the laying head and is rotating about the axis defined by the inlet wire.

The pipe optimization problem has been studied at the Morgan Construction Company since the early 1970's. In 1999, Morgan sponsored a team of undergraduate students supervised by Prof. B. Vernescu at the Worcester Polytechnic Institute in a study of pipe optimization, [1]. Their approach to pipe optimization was to assume a path for the wire, compute the resulting force on the pipe, and from this force compute the wear. In their study, the wear, $w(s)$, where s denotes arc length along the wire is defined as

$$w(s) = \sqrt{(\vec{F} \cdot \vec{n})^2 + (\vec{F} \cdot \vec{b})^2}. \tag{1}$$

Here, $\vec{F}$ is the force of the wire on the pipe, $\vec{n}$ is the normal to the wire axis, and $\vec{b}$ is the binormal. Optimization was then performed by computing the L^p norm of the wear distribution and adjusting the wire path in order to reduce this norm. This approach had two notable successes. First, assuming the wire followed the centerline of the current Morgan pipe design, the "two-humped" wear distribution shown in Figure **??** was produced. This agreed with experimental wear measurements performed at Morgan. Next, implementing the optimization process numerically resulted in a new pipe design with the wear distribution shown in Figure **??**. (BOGDAN, COULD YOU PLEASE SEND ME EPS FILES OF THESE TWO PLOTS?) This is an improvement over the current pipe design; the maximum wear is reduced and the wear is more evenly distributed. On the downside, the tail of the distribution is extremely irregular; it is speculated that this is the result of numerical instabilities.

In this study, we take a different but complementary approach to pipe optimization. Rather than prescribing the path, computing the wear, and then adjusting the path to reduce the wear, we attempt to impose the desired wear distribution and find a pipe which wears in this manner. In particular, we ask

> *If a force of constant magnitude is to be applied everywhere along the wire, along what shape path must we apply it in order to satisfy the inlet and outlet conditions?*

The constant force hypothesis is motivated by the intuition that distributing the wear as evenly as possible, which corresponds to a constant force, is in some sense "optimal."

We begin in the next section with a two-dimensional toy problem designed to illustrate our approach. In this case, the problem may be solved analytically. The solution aids in developing intuition. Specifically the dependence of the solution on parameters describing the frictional force between the pipe and wire is revealed. Most interestingly, the problem is seen to be an *eigenvalue* problem. In Section 3, we formulate a full three-dimensional version of the pipe optimization problem. The equations are recast in non-dimensional form. We discuss the boundary conditions and the nature of the eigenvalue problem for the three dimensional case. Preliminary numerical results are described. Finally, in Section 4, we end with a discussion of our results to date and recommendations for future work.

2 A 2-d Toy Problem

In this section we consider the idealized problem of changing the direction of motion of a wire moving in a plane. We attempt to apply a force of constant magnitude along the path and solve for the path which will allow us to satisfy inlet and outlet conditions.

2.1 The Governing Equations

Consider the system shown in Figure 3. At the left, i.e., at $z = 0$ we

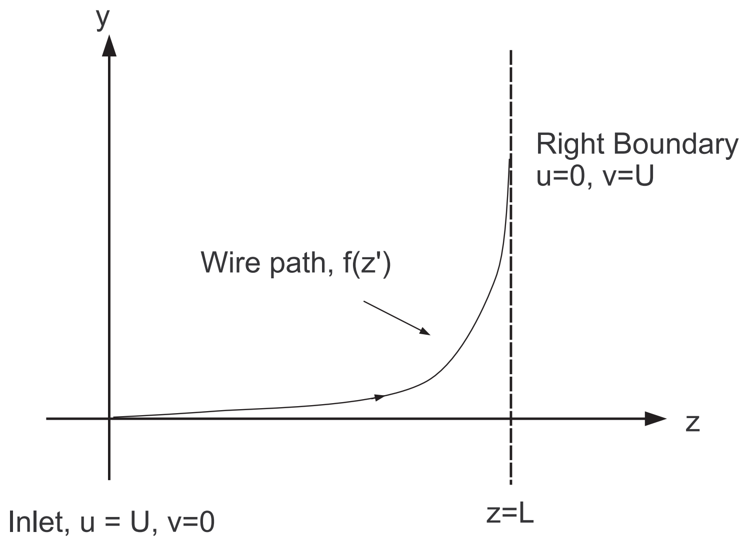

Figure 3: Setup of the 2-d problem.

imagine that a wire with velocity U, is travelling along the z axis. At $z = L$, we want the motion of the wire to be perpendicular to the z axis. Assuming we are to apply a force of the same constant magnitude at each point along the wire we ask what path the wire must follow. If we knew such a path, a pipe could then be constructed around that path and wear would occur in an even manner.

Since mass is conserved, the components of the velocity vector, $\vec{u} = (u, v)$, must satisfy

$$u^2 + v^2 = U^2. \tag{2}$$

If the shape of the path is described by the function $y = f(z)$, the velocity vector at each point must point in the direction of the tangent to the path, i.e., parallel to $\vec{t}$, where

$$\vec{t} = \frac{(1, f')}{\sqrt{1 + (f')^2}}, \tag{3}$$

or perpendicular to the normal $\vec{n}$, where

$$\vec{n} = \frac{(-f', 1)}{\sqrt{1 + (f')^2}}. \tag{4}$$

That is, we require $\vec{u} \cdot \vec{n} = 0$ or

$$\frac{v}{u} = f'. \tag{5}$$

Here, primes denote differentiation with respect to z. Next, we apply a force balance in the z and y directions. This yields

$$\rho u u' = F_z + \frac{T'}{1+(f')^2} - \frac{T f' f''}{\left(1+(f')^2\right)^2} \tag{6}$$

$$\rho u v' = F_y + \frac{f' T'}{1+(f')^2} + \frac{T f''}{\left(1+(f')^2\right)^2}. \tag{7}$$

Here, ρ is the linear density of the wire, T is the unknown tension along the wire, and $\vec{F} = (F_z, F_y)$ is the vector of applied forces. We assume that the applied force *normal* to the wire has constant magnitude. That is,

$$\vec{F} \cdot \vec{n} = F. \tag{8}$$

We assume the static force *tangent* to the wire is proportional to the normal force. That is

$$\vec{F} \cdot \vec{t} = \mu F, \tag{9}$$

where μ is the coefficient of sliding friction. We may solve for the components of $\vec{F}$ in terms of μ and F to find

$$F_z = \frac{F(\mu - f')}{\sqrt{1+(f')^2}} \tag{10}$$

$$F_y = \frac{F(1+\mu f')}{\sqrt{1+(f')^2}}. \tag{11}$$

To satisfy the input and output criteria, we need to impose the appropriate boundary conditions. The conditions on f, u, and v are

$$u(0) = U, \qquad v(0) = 0, \qquad f(0) = 0, \tag{12}$$

at $z = 0$, and

$$u(L) = 0, \qquad v(L) = U, \qquad f'(L) = \infty, \tag{13}$$

at $z = L$. On the unknown tension T, we impose the end condition

$$T(L) = 0. \tag{14}$$

That is, the wire is not being "pulled" at the right end. A count of unknowns and equations shows that this system is over-determined. Thus, we expect a solution to exist for only special values of the parameters, i.e. the problem is an eigenvalue problem. This makes physical sense as we expect to find solution for only special values of the constant F which plays the role of an eigenvalue for this problem and must be determined as part of the solution.

2.2 Scaling

It is convenient to recast our problem in terms of dimensionless variables. We introduce the scalings

$$\hat{z} = \frac{z}{L}, \qquad \hat{f} = \frac{f}{L}, \qquad \hat{u} = \frac{u}{U}, \qquad \hat{v} = \frac{v}{U}, \qquad \hat{T} = \frac{T}{A}, \tag{15}$$

into our governing equations and then drop the hats for convenience. The resulting dimensionless system is

$$u^2 + v^2 = 1 \tag{16}$$

$$\frac{v}{u} = f' \tag{17}$$

$$uu' = \lambda \left(\frac{-f' + \mu}{\sqrt{1 + (f')^2}} \right) + \frac{T'}{\sqrt{1 + (f')^2}} - \frac{Tf'f''}{\left(1 + (f')^2\right)^2} \tag{18}$$

$$uv' = \lambda \left(\frac{1 + \mu f'}{\sqrt{1 + (f')^2}} \right) + \frac{f'T'}{\sqrt{1 + (f')^2}} + \frac{Tf''}{\left(1 + (f')^2\right)^2} \tag{19}$$

with scaled boundary conditions

$$u(0) = 1, \qquad v(0) = 0, \qquad f(0) = 0, \tag{20}$$

$$u(1) = 0, \qquad v(1) = 1, \qquad f'(1) = \infty, \qquad T(1) = 0. \tag{21}$$

The dimensionless parameter λ is given by

$$\lambda = \frac{FL}{\rho U^2}. \tag{22}$$

Note, λ is the eigenvalue for the dimensionless problem.

2.3 The Solution

It is straightforward to reduce the problem given by (16), (17), (18) and (19) with boundary conditions (20) and (21) to quadratures. First, note that (16) may be differentiated to give

$$uu' + vv' = 0, \tag{23}$$

or

$$uu' + f'uv' = 0, \tag{24}$$

using (17). We may now eliminate uu' and vu' using (18) and (19) and simplify to give

$$T' = -\mu\lambda. \tag{25}$$

This simple equation reveals the expected result that the rate of decrease of the tension is equal to the coefficient of friction times the constant (scaled) normal force. Integration and the boundary condition for T give

$$T(z) = \mu\lambda(1 - z). \tag{26}$$

Further progress can be made if we differentiate (17) and use (16) to obtain

$$uv' = uu'f' + \frac{f''}{1 + (f')^2}. \tag{27}$$

Again, we may eliminate uu' and vu' using (18) and (19) and simplify to find

$$\frac{\lambda}{1 - T(z)} = \frac{f''}{\big(1 + (f')^2\big)^{3/2}}. \tag{28}$$

We may now use (26) to eliminate $T(z)$ and integrate to obtain

$$1 + \frac{1}{\mu}\ln\big(1 - \mu\lambda(1 - z)\big) = \frac{f'}{\sqrt{1 + (f')^2}}. \tag{29}$$

We note that the constant of integration is chosen so that $f'(1) = \infty$. We also note that $u(0) = 1$ and $v(0) = 0$ imply $f'(0) = 0$ so that

$$1 + \frac{1}{\mu}\ln(1 - \mu\lambda) = 0. \tag{30}$$

This is a constraint on the dimensionless force, an eigenvalue, which may be written explicitly in the form

$$\lambda = \frac{1 - e^{-\mu}}{\mu}. \tag{31}$$

Finally, the shape of the path is obtained by solving for f' in (29) and integrating. This results in the formula

$$f(z) = \int_0^z \frac{g(z)}{\sqrt{1 - g(z)^2}} dz, \tag{32}$$

where

$$g(z) = 1 + \frac{1}{\mu} \ln\bigl(1 - \mu\lambda(1 - z)\bigr) \tag{33}$$

and λ is given in (31). The shape depends only on the choice for the coefficient of friction μ.

In Figure 4, we plot the shape of the path for various values of μ. Note that when $\mu = 0$ the solution reduces to a semi-circular arc.

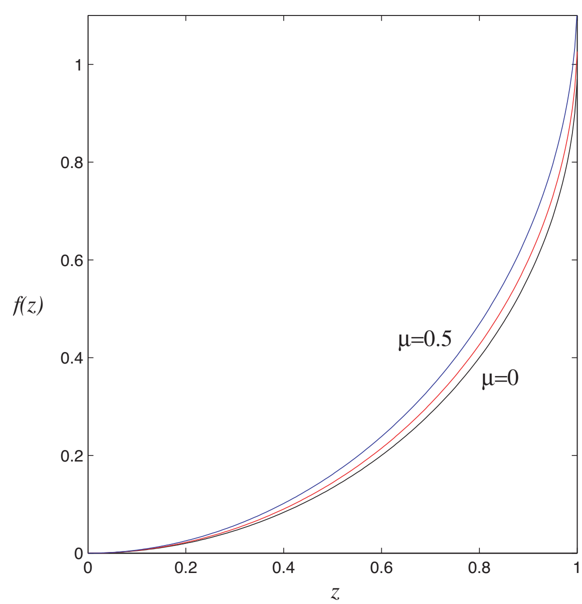

Figure 4: Path of the wire for $\mu = 0$, 0.2 and 0.5

3 The 3-d Model

In this section we consider the full three dimensional version of the problem. As discussed above, we will attempt to apply a force of constant magnitude along a path constructed to satisfy the inlet and outlet conditions. We seek to find the path. Our experience with the 2-d problem suggests that we will arrive at an eigenvalue problem. This is in fact the case.

3.1 The Governing Equations

Our coordinate system is shown in Figure 5. Note that the path is defined

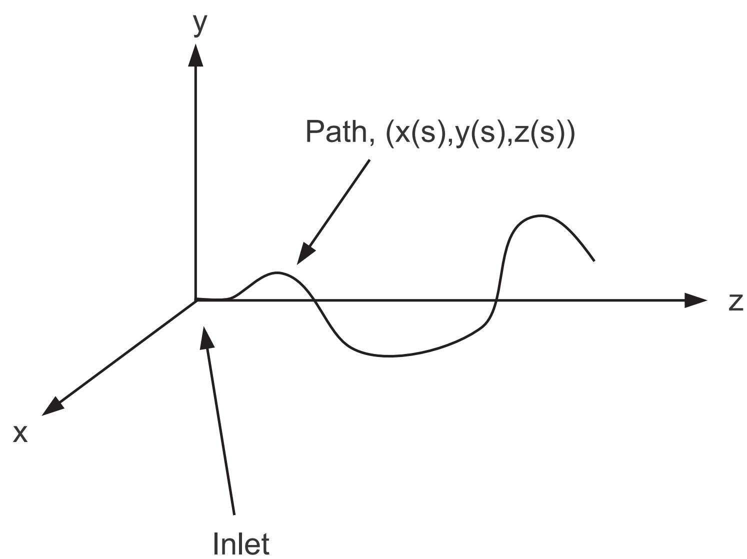

Figure 5: Setup of the 3-d problem.

by the space curve $(x(s), y(s), z(s))$ where s is the arc length along the path. We imagine the wire enters at the origin and is travelling along the z axis with velocity U. At $z = L$ we want the wire to possess no further velocity in the z direction and to be stationary upon exiting the pipe as viewed by an observer in the laboratory frame. Additionally, the observer in the laboratory frame will see the right end of the *pipe* moving with angular speed ω. The axis of rotation is the z axis. As in the 2-d study, if we knew such a path, a pipe could then be constructed around that path and wear would occur with a uniform distribution.

Any space curve must satisfy certain geometric constraints. First, if we define intrinsic Frenet coordinates along the path, the Frenet triad must satisfy the Serret-Frenet equations

$$\frac{d\vec{t}}{ds} = k\vec{n} \tag{34}$$

$$\frac{d\vec{n}}{ds} = -k\vec{t} + \tau\vec{b} \tag{35}$$

$$\frac{d\vec{b}}{ds} = -\tau\vec{n}. \tag{36}$$

Here, $\vec{n}$ is the unit normal to the path, $\vec{b}$ the unit bi-normal, and $\vec{t}$ the unit tangent vector. The proportionality constants k and τ are the curvature and torsion, respectively. Also, the definition of arc-length implies that

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1. \tag{37}$$

In addition to geometric constraints, the motion of the wire must satisfy conservation of momentum. That is,

$$\rho\vec{a} = \vec{F} + \frac{dT}{ds}\vec{t} + kT\vec{n}. \tag{38}$$

This is the familiar force equals mass times acceleration. As in the 2-d case, T is tension along the wire and $\vec{F}$ is the external applied force. Note that k is related to $(x(s), y(s), z(s))$ via

$$k^2 = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2. \tag{39}$$

The components of the velocity vector $\vec{v} = (u, v, w)$ are related to those of the position vector $R = (x(s), y(s), z(s))$ and the acceleration vector through

$$\vec{v} = U\vec{t} + (0, 0, \omega) \times \vec{R}. \tag{40}$$

Here we have assume a rotation about the z axis with angular speed ω. For the applied force, we again assume that the force of static friction tangent to the wire is proportional to the normal force. Unlike the 2-d case, however, it is not sufficient to simply prescribe that the magnitude of the normal force be constant. Since we are in 3-d, we must also specify the angle that the applied normal force makes with the unit normal to the wire. We specify this angle through the function $\phi(s)$ and the applied force via

$$\vec{F} = F\cos(\phi(s))\vec{n} + F\sin(\phi(s))\vec{b} + \mu F\vec{t}. \tag{41}$$

Note that with this definition for $\vec{F}$, the tangential force is proportional to F and F is the constant magnitude of the normal force. Also note that $\phi(s)$ is a function we prescribe. That is, $\phi(s)$ introduces an extra degree of freedom into the three-dimensional problem. Ultimately, one wishes to optimize over all admissible $\phi(s)$.

We must prescribe the appropriate boundary conditions to satisfy input and output conditions. Since we assume the wire enters at the origin of our coordinate system and is moving along the z axis initially, we impose

$$x(0) = y(0) = z(0) = 0, \qquad x'(0) = y'(0) = 0. \tag{42}$$

At the end of the wire, which occurs at $s = \ell$, where ℓ is a priori unknown, we impose

$$z(\ell) = L, \quad z'(\ell) = T(\ell) = x'(\ell) - \omega y(\ell) = y'(\ell) + \omega x(\ell) = 0. \tag{43}$$

The two boundary conditions involving ω imply that the end of the wire is stationary as seen by a laboratory observer.

3.2 Scaling and Simplification

It is convenient to recast our problem in dimensionless form and to eliminate the components of the velocity vector in favor the appropriate derivatives of the components of the position vector. We introduce the dimensionless variables

$$\hat{x} = \frac{x}{L}, \quad \hat{y} = \frac{y}{L}, \quad \hat{z} = \frac{z}{L}, \quad \hat{s} = \frac{s}{L}, \quad \hat{T} = \frac{T}{\rho U^2}, \quad \hat{k} = Lk \tag{44}$$

into our governing equations and subsequently drop the hats for notational convenience. Eliminating the components of the velocity vector $\vec{v}$, we arrive at the dimensionless equations

$$T' + \mu\lambda = -\Omega^2(xx' + yy'), \tag{45}$$

$$-\frac{2\Omega}{k}z'' + \frac{\Omega^2}{k}\big(-x(y'z''-z'y'') - y(-x'z''+z'x'')\big) = \lambda\sin(\phi(s)), \tag{46}$$

$$k + \frac{2\Omega}{k}(x'y''-y'x'') - \frac{\Omega^2}{k}(xx''+yy'') = \lambda\cos(\phi(s)) + Tk, \tag{47}$$

$$k^2 = (x'')^2 + (y'')^2 + (z'')^2, \tag{48}$$

$$(x')^2 + (y')^2 + (z')^2 = 1. \tag{49}$$

Equations (45), (46) and (47) represent the components of the force balance (38) in the directions of the tangent, binormal and normal, respectively. Equations (48) and (49) are the dimensionless versions of (39) and (37), respectively. It should be noted that primes now denote differentiation with respect to (dimensionless) arclength s (and not z as in the 2-d problem). The boundary conditions at $s = 0$ become

$$x(0) = y(0) = z(0) = 0, \quad x'(0) = y'(0) = 0, \tag{50}$$

while the conditions at $s = \alpha$, where $\alpha = \ell/L$, are

$$z(\alpha) = 1, \quad z'(\alpha) = T(\alpha) = x'(\alpha) - \Omega y(\alpha) = y'(\alpha) + \Omega x(\alpha) = 0. \tag{51}$$

The equations above are to be solved for (x, y, z, T, k) as functions of s for a given choice for $\phi(s)$, μ and Ω. As in the 2-d problem, the eigenvalue λ must also be determined as part of the problem.

3.3 Local Analysis at the Inlet

Analytical progress towards a solution of the eigenvalue problem is limited to a local analysis of the solution near $s = 0$. A full solution requires a numerical treatment and this be discussed in the next section.

Near $s = 0$, we assume that $x(s)$, $y(s)$, $z(s)$ and $T(s)$ have the form

$$x(s) \sim \frac{x_2 s^2}{2} + \ldots, \quad y(s) \sim \frac{y_2 s^2}{2} + \ldots, \quad z(s) \sim s + \frac{z_3 s^3}{6} + \ldots, \tag{52}$$

and

$$T(s) \sim T_0 - \mu\lambda s + \frac{T_4 s^4}{24} + \ldots, \tag{53}$$

where x_2, y_2, z_3, T_0 and T_4 are constants. These forms already reflect the boundary conditions at $s = 0$ and the anticipated behavior for $T(s)$ in the equations. We now proceed by substituting these forms into the various equations. First, we note that (48) gives

$$k(s)^2 \sim x_2^2 + y_2^2 + \ldots. \tag{54}$$

Let us define $k_0 = \sqrt{x_2^2 + y_2^2}$. From the derivative of (49), we find

$$z_3 = -x_2^2 - y_2^2 = -k_0^2. \tag{55}$$

Next, we consider (45) which gives

$$T_4 = -3\Omega^2 k_0^2. \tag{56}$$

Finally, from (46) and (47), we obtain

$$2\Omega k_0 s + \ldots \sim \lambda \sin(\phi(s)), \qquad (1 - T_0)k_0 + \ldots \sim \lambda \cos(\phi(s)), \tag{57}$$

which suggests that $\phi(s) \sim \phi_1 s + \ldots$, and thus

$$\phi_1 = \frac{2\Omega}{1 - T_0} \quad \text{and} \quad k_0 = \frac{\lambda}{1 - T_0} \tag{58}$$

The local analysis determines how the path of the wire behaves near $s = 0$, and gives us useful information for the numerical procedure discussed in the next section.

3.4 Numerical Method and Solutions

We now turn to a discussion of a numerical method for the eigenvalue problem. Our basic approach is to construct a shooting method that integrates the equations numerically from $s = 0$ to $s = \alpha$, using the local analysis to near $s = 0$ to begin the integration. A number of parameters and needed to preform the integration, including the eigenvalue λ, and these adjusted iteratively until the boundary conditions at $s = \alpha$ are satisfied.

We begin by writing the equations as a system of first order equations. Let

$$\mathbf{u}(s) = (x, y, z, x', y', z', T). \tag{59}$$

By definition, we have

$$u_1' = u_4, \qquad u_2' = u_5, \qquad u_3' = u_6. \tag{60}$$

Equations (45), (46) and (47) become

$$u_7' + \mu\lambda = -\Omega^2(u_1u_4 + u_2u_5), \tag{61}$$

$$-\frac{2\Omega}{k}u_6' + \frac{\Omega^2}{k}\big(-u_1(u_5u_6' - u_6u_5') - u_2(-u_4u_6' + u_6u_4')\big) = \lambda\sin(\phi(s)), \tag{62}$$

$$k + \frac{2\Omega}{k}(u_4u_5' - u_5u_4') - \frac{\Omega^2}{k}(u_1u_4' + u_2u_5') = \lambda\cos(\phi(s)) + ku_7, \tag{63}$$

respectively, with k given by

$$k^2 = (u_4')^2 + (u_5')^2 + (u_6')^2. \tag{64}$$

Upon differentiation, (49) becomes

$$u_4u_4' + u_5u_5' + u_6u_6' = 0. \tag{65}$$

We observe that the three equations in (60) along with (61), (62), (63) and (65) form a system of seven first order equations of the form

$$G\big(\mathbf{u}(s), \mathbf{u}'(s); \lambda, \mu, \Omega, \phi(s)\big) = 0. \tag{66}$$

Boundary conditions for the system of equations at $s = \delta$ may be taken using the local solution near $s = 0$ assuming that δ is small. This is done to avoid the singularity in the equations at $s = 0$, which occurs because the equations are unchanged if x and y (and their derivatives) are interchanged. To set the coordinates x and y, we take $y_2 = 0$ in the local solution (i.e., in the definition for k_0). With this choice, we have

$$u_1 = \frac{k_0\delta^2}{2}, \quad u_2 = 0, \quad u_3 = \delta - \frac{k_0^2\delta^3}{6}, \tag{67}$$

$$u_4 = k_0\delta, \quad u_5 = 0, \quad u_6 = 1 - \frac{k_0^2\delta^2}{2}, \quad u_7 = T_0 - \mu\lambda\delta - \frac{\Omega^2k_0^2\delta^4}{8}, \tag{68}$$

with $k_0 = \lambda/(1 - T_0)$. Also, according to our local solution, we must choose a function $\phi(s)$ such that

$$\phi'(0) = \phi_1 = \frac{2\Omega}{1 - T_0}. \tag{69}$$

Assuming that δ and $\phi(s)$ are given, we may choose provisional values for λ, μ, Ω and T_0 and integrate (66) for $s > \delta$. The solution of this initial-value problem would have the form

$$\mathbf{u} = \mathbf{u}(s; \lambda, \mu, \Omega, T_0), \tag{70}$$

i.e., $\mathbf{u}$ depends on s and on the chosen parameters. At $s = \alpha$, the solution of the initial-value problem must satisfy the five boundary conditions given in (51). These boundary conditions may be written in the form $\mathcal{F} = 0$, where

$$\mathcal{F}(\alpha, \lambda, \mu, \Omega, T_0) = \begin{pmatrix} u_3(\alpha; \lambda, \mu, \Omega, T_0) - 1 \\ u_4(\alpha; \lambda, \mu, \Omega, T_0) \\ u_7(\alpha; \lambda, \mu, \Omega, T_0) \\ u_4(\alpha; \lambda, \mu, \Omega, T_0) - \Omega u_2(\alpha; \lambda, \mu, \Omega, T_0) \\ u_5(\alpha; \lambda, \mu, \Omega, T_0) + \Omega u_1(\alpha; \lambda, \mu, \Omega, T_0) \end{pmatrix} \tag{71}$$

The vector equation $\mathcal{F} = 0$ is a system of five nonlinear algebraic equations that must be solved for α, λ, μ, Ω and T_0. A numerical integration of the system of nonlinear ODEs in (66) is required to evaluate $\mathcal{F}$ for a given choice for $(\alpha, \lambda, \mu, \Omega, T_0)$, and a numerical method of iteration, Newton's method say, is needed to find the choice of parameters that solve $\mathcal{F} = 0$. This is the essential idea of our shooting method, a more detailed discussion of shooting methods in general may be found in [2].

Assuming that $(\alpha, \lambda, \mu, \Omega, T_0)$ may be found that satisfy $\mathcal{F} = 0$, the task then would be to adjust $\phi(s)$ (with $\phi'(0) = 2\Omega/(1-T_0)$) so that the eigenvalue λ is minimized. This would correspond to a wire path that minimizes wear.

It should be noted, however, that the algebraic equations $\mathcal{F} = 0$ are nonlinear and may not have a solution for all functions $\phi(s)$. In our numerical experiments, we have chosen the simple function

$$\phi(s) = \frac{\pi}{2} \tanh\left(\frac{2\Omega}{1 - T_0} \frac{2s}{\pi}\right) \tag{72}$$

which satisfies $\phi'(0) = 2\Omega/(1 - T_0)$ and $\phi \to \pi/2$ as s becomes large. The asymptotic behavior as s becomes large agrees with the results given in [1]. With this choice for $\phi(s)$, we have been unable to find $(\alpha, \lambda, \mu, \Omega, T_0)$ so that $\mathcal{F} = 0$. We are able to obtain a path for the wire which satisfies the first three components of $\mathcal{F} = 0$, but the velocity at the exit as given by the last two components of $\mathcal{F}$ is not zero. Further work involving a method of iteration on the function $\phi(s)$ is needed to obtain solutions with $\mathcal{F} = 0$.

Figure 6 shows the path of the wire for the parameters

$$\lambda = 0.4996, \quad \mu = 0.2, \quad \Omega = 1.956, \quad T_0 = 0.5. \qquad (73)$$

For this calculation, the equations are integrated numerically using the backward Euler method (see [2]) from $s = \delta = 0.005$ to $\alpha = 1.795$ where $z = u_3 = 1$. Figures 7(a) and (b) show the behavior of the tension T and curvature k, respectively, along the path as functions of z. We note that T decreases monotonically to zero along the path. The magnitude of the velocity at $z = 1$ for this path is 0.2669, and further adjustment of the parameters for the chosen $\phi(s)$ does not reduce this number significantly.

4 Discussion

We have introduced a new approach for optimizing the design of a laying pipe. It is important to note that the approach introduced here differs significantly from the approach in [1]. In [1], the path for the wire was assumed, the force on the pipe computed from this path, and the wear computed from this force. The wire path was then adjusted to minimize the wear. In the approach outlined in this report, we constrain the wire path to be one such that the force on the pipe is constant, and then attempt to compute a wire path with this property. The conjecture is that if the force is constant, the pipe will wear evenly. The 2-d model of this approach was solved completely. The 3-d version was formulated and a numerical approach to the solution developed. A complete numerical study following this approach should yield an optimal pipe design.

5 Acknowledgements

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References

[1] B.P. Hogan and C. Lee, *Shape Optimization in Laying Pipe Design*, CIMS Report, Worcester Polytechnic Institute, (1999).

[2] Uri M. Ascher and Linda R. Petzold, *Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations*, SIAM, Philadelphia, 1998.

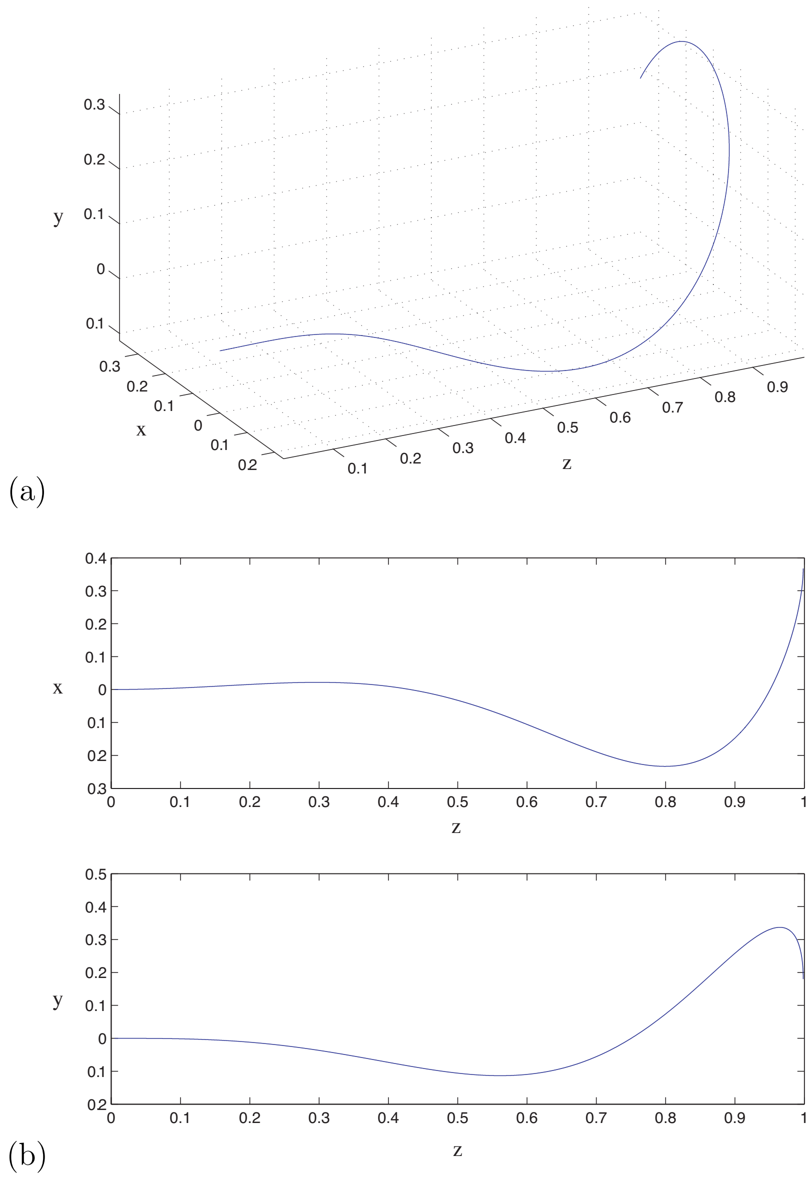
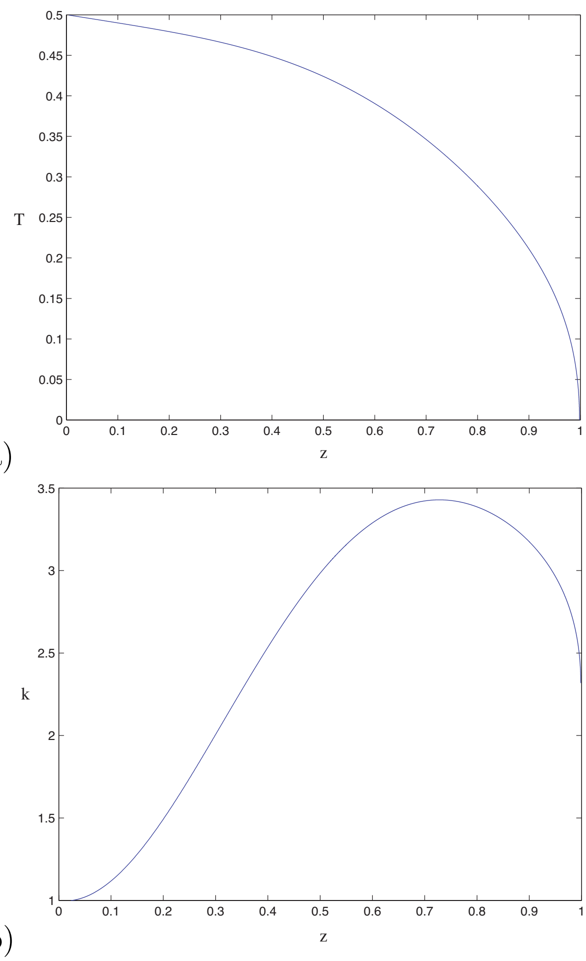


Figure 6: Path of the wire: (a) three-dimensional view; (b) projections in the planes $y = 0$ and $x = 0$.



(a)

(b)

Figure 7: Behavior of the tension (a) and curvature (b) along the wire.